

# Moutard type transformation for matrix generalized analytic functions and gauge transformations \*

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Considerable progress in the theory of Darboux-Moutard type transformations for two-dimensional linear differential systems with applications to geometry, spectral theory, and soliton equations has been achieved recently, see, e.g., [1, 2, 3, 4]. In the present note we derive such a transformation for the matrix generalized function system

$$\partial_{\bar{z}}\Psi + A\Psi + B\bar{\Psi} = 0, \quad (1)$$

where  $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$ , the coefficients  $A$  and  $B$  and solutions  $\Psi$  are  $(N \times N)$ -matrix functions on  $D$ , with  $D$  an open simply connected domain in  $\mathbb{C}$ . In particular, this generalizes the transformation for  $N = 1$  found in [4] with  $A = 0$ . In addition, we show that the Moutard type transformation for system (1) with  $B = 0$  is equivalent to a gauge transformation for the connection  $\nabla_{\bar{z}} = \partial_{\bar{z}} + A$ . In turn, our studies show that the Moutard type transformation for system (1) with  $A = 0$  can be treated as a proper analog of the forementioned gauge transformation.

As for  $N = 1$ , system (1) is reduced to the system

$$\partial_{\bar{z}}\Psi + B\bar{\Psi} = 0, \quad (2)$$

i.e. to system (1) with  $A = 0$ , by the gauge transformation

$$\Psi \rightarrow \tilde{\Psi} = g^{-1}\Psi, \quad B \rightarrow \tilde{B} = g^{-1}Bg, \quad \partial_{\bar{z}}g + Ag = 0, \quad \det g \neq 0.$$

We say that the system

$$\partial_z\Psi^+ - \bar{\Psi}^+B = 0 \quad (3)$$

is conjugate to system (2) (see [5] for a similar definition for  $N = 1$ ).

We have the following result.

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\*The work was done during the visit of the second author (I.A.T.) to Centre de Mathématiques Appliquées of École Polytechnique in July 2016 and was supported by RSF (grant 14-11-00441).

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**Theorem 1** *Systems (2) and (3) are covariant, i.e. mapped into the systems of the same type, with respect to the Moutard type transformation*

$$\begin{aligned}\Psi &\rightarrow \tilde{\Psi} = \Psi - F \omega_{F,F+}^{-1} \omega_{\Psi,F+}, \\ \Psi^+ &\rightarrow \tilde{\Psi}^+ = \Psi^+ - \omega_{F,\Psi^+} \omega_{F,F+}^{-1} F^+, \\ B &\rightarrow \tilde{B} = B + F \omega_{F,F+}^{-1} F^+, \end{aligned} \quad (4)$$

where  $F$  and  $F^+$  are arbitrary fixed solutions of (2) and (3), respectively,

$$\partial_{\bar{z}} \omega_{\Phi,\Phi^+} = \Phi^+ \bar{\Phi}, \quad \text{Re } \omega_{\Phi,\Phi^+} = 0, \quad (5)$$

for  $\Phi$  and  $\Phi^+$  meeting equations (2) and (3), and  $\det \omega_{F,F+} \neq 0$ .

For finding  $\omega_{\Phi,\Phi^+}$  satisfying (5) we use also that  $\partial_z \omega_{\Phi,\Phi^+} = -\bar{\Phi}^+ \Phi$ . In addition, our definition of  $\omega_{\Phi,\Phi^+}$  is self-consistent up to a pure imaginary matrix integration constant in view of the identity  $\partial_z \Phi^+ \bar{\Phi} = -\partial_{\bar{z}} \bar{\Phi}^+ \Phi$ . The latter equality follows from systems (2) and (3) for  $\Phi$  and  $\Phi^+$ , respectively. We recall that the domain  $D$  is simply connected.

Given  $\omega_{F,F+}$ ,  $\omega_{\Psi,F+}$ , and  $\omega_{F,\Psi^+}$ , Theorem 1 is proved by straightforward computations.

In addition, for the system

$$\partial_{\bar{z}} \Psi + A \Psi = 0, \quad (6)$$

i.e., for system (1) with  $B = 0$ , the following result also holds.

**Proposition 1** *System (6) is covariant under the following Moutard type transformation*

$$\Psi \rightarrow \tilde{\Psi} = \Psi - F \hat{\omega}_{F,F+}^{-1} \hat{\omega}_{\Psi,F+}, \quad A \rightarrow \tilde{A} = A + F \hat{\omega}_{F,F+}^{-1} F^+, \quad (7)$$

where  $F$  is an arbitrary fixed solution of (6),  $F^+$  is an arbitrary fixed matrix function,

$$\partial_{\bar{z}} \hat{\omega}_{\Phi,F+} = F^+ \Phi \quad (8)$$

for any matrix function  $\Phi$ , and  $\det \hat{\omega}_{F,F+} \neq 0$ .

Equations (7) and (8) are analogs of equations (4) and (5). However, in difference with (5), we do not require that the matrix functions  $\hat{\omega}_{F,F+}$  would be pure imaginary. Equation (8) is solvable for  $\hat{\omega}_{F,F+}$  and Proposition 1 is proved by straightforward computations.

REMARK. Let  $A, \tilde{A}, \Psi, F, F^+$ , and  $\hat{\omega}_{\Phi,F+}$  be the same as in Proposition 1. Let

$$g = 1 - F \hat{\omega}_{F,F+}^{-1} \Lambda, \quad \Lambda_{\bar{z}} = \Lambda A + F^+.$$

Then

$$\partial_{\bar{z}}(g\Psi) + \tilde{A}(g\Psi) = 0.$$

It is proved by straightforward computations and it shows that for invertible  $g$  the transformation  $A \rightarrow \tilde{A}$  reduces to a gauge transform.

## References

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